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LETTER TO THE EDITOR

Conditions for the existence of oscillations in the distribution of the vibrational quanta of squeezed states

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Abstract. The conditions for the existence of oscillations in the distribution of the vibrational quanta for the general case of the time-evolving squeezed state (which does not remain a minimum-uncertainty state) of the one-dimensional harmonic oscillator are investigated. To aid this, new parametrization of squeezed states is introduced. It is found that in addition to the usual conditions which produce oscillations, the value of the phase ζ of the complex parameter z (which is the argument of the Hermite polynomial H_n appearing in the expression for the expansion coefficient a_n of the squeezed state in the number state basis) must have a value in the vicinity of $\zeta = 0$ or $\zeta = \pi$ ($|\mathrm{Im}\{z\}| \ll |z|$). It is shown how this necessary condition results from the explicit expression for $|H_n(z)|^2$.

Squeezed light [1–4] has a lower quantum noise in one of its field quadratures than the coherent light emitted by an (ideal) laser. The low intrinsic noise of squeezed light has led to suggestions for its use, among other things, for detecting gravitational waves [5] and in optical telecommunication systems [6]. Inasmuch as the quantized electromagnetic field is equivalent to a collection of harmonic oscillators, a mathematical description of squeezed states is rooted in the quantum mechanics of the one-dimensional harmonic oscillator. The purpose of this letter is, firstly, to provide a different parametrization of squeezed states of the one-dimensional harmonic oscillator. This parametrization turns out to be, as we shall see, more convenient for investigating the conditions for the existence of oscillations in the probability distribution of the vibrational quanta, to which we turn our attention in the second part of the letter. These oscillations, and the conditions which produce them, were, for the minimum-uncertainty squeezed states, first discussed in 1985 by Wheeler [7] and later by Schleich and Wheeler [8–10]. They found that, loosely speaking, a minimum-uncertainty squeezed state exhibits oscillations in the distribution of the vibrational quanta for large squeezing and sufficiently large vibrational quantum numbers. The physical principle behind oscillations is interference in phase space [9, 10]. Here we consider the oscillations for the more general case of the time-evolving squeezed state (which does not remain a minimum-uncertainty state) and find that, additionally, the value of the phase ζ of the complex parameter z (which is the argument of the Hermite polynomial $H_n(z)$ appearing in the expression for the expansion coefficient a_n of the squeezed state in the number state basis) is important. In order to obtain the oscillations it is necessary that the value of the phase be in the vicinity of $\zeta = 0$ or $\zeta = \pi$ ($|\mathrm{Im}\{z\}| \ll |z|$). The generic case is depicted in figure 1. Finally, we demonstrate how this necessary condition results from the explicit expression for $|H_n(z)|^2$.

First, we summarize briefly relevant results and establish notation. The Schrödinger picture is used throughout. The general solution of the time-dependent Schrödinger equation for the harmonic oscillator reads

$$\Psi(x, t) = \sum_{n=0}^{\infty} a_n \exp\left(-\frac{i}{\hbar} E_n t\right) u_n(x). \tag{1}$$

Here $u_n(x)$ denotes the harmonic oscillator eigenfunction corresponding to the nth energy level E_n [11]. When we take [3, 12, 13]

$$a_n \equiv N\left(\frac{s^n}{2^n n!}\right)^{1/2} H_n(z) \tag{2}$$

we obtain the squeezed state $\Psi_{sz}(x, t)$. Here, H_n is the nth Hermite polynomial, while $s \equiv s_1 + is_2 = |s| e^{i\theta}$ and $z \equiv z_1 + iz_2 = |z| e^{i\zeta}$ are the complex parameters used to specify the squeezed state. There are altogether four independent real parameters s_1, s_2, z_1 and z_2 (or, equivalently $|s|$, θ, $|z|$ and ζ). In order to ensure the convergence of the series (1) we must take $|s| < 1$. The parametrization introduced here differs from the one usually used. We found that it leads to a somewhat simpler parametric dependence of the oscillations in the probability distribution of the vibrational quanta. The complex quantity N, appearing in (2), is determined by the normalization condition $\Sigma |a_n|^2 = 1$:

$$N = (1 - |s|^2)^{1/4} \exp\left[-\frac{|s|(|z|^2 - |s| z^2)}{1 - |s|^2}\right]. \tag{3}$$

Explicitly, the time-dependent wavefunction representing the squeezed state is

$$\Psi_{sz}(x, t) = \gamma \exp(-\delta(x - \varepsilon)^2) \tag{4}$$

where

$$\gamma \equiv \frac{N}{(2\pi)^{1/4}(\sigma C_-)^{1/2}} \exp\left(-\frac{i\omega t}{2} + \frac{sz^2}{C_+} e^{-2i\omega t}\right) \tag{5}$$

$$\delta \equiv \frac{C_+}{4\sigma^2 C_-} \tag{6}$$

$$\varepsilon \equiv \frac{2^{3/2}\sigma s^{1/2} z}{C_+} e^{-i\omega t} = \frac{2^{3/2}\sigma |s|^{1/2} |z|}{C_+} e^{-i\Phi_+} \tag{7}$$

with

$$C_\pm = 1 \pm s\, e^{-2i\omega t} \qquad \Phi_\pm \equiv \theta/2 \pm \zeta - \omega t. \tag{8}$$

The squeezed states have a number of interesting properties. They are not mutually orthogonal. The set $\{\Psi_{sz}(x, t)\}$ is (over)complete. The probability density $|\Psi_{sz}(x, t)|^2$ is normalized Gaussian with halfwidth changing periodically in time and depending only on the squeeze parameter s (it is independent of the other complex parameter z). The general squeezed state $\Psi_{sz}(x, t)$ in time is not a minimum-uncertainty state; nevertheless, every full period it becomes four times a minimum-uncertainty state. The uncertainties δx and δp oscillate in width, out of phase; the uncertainty in one of the two dynamically conjugate variables x and p can become less than the corresponding one in a coherent state. When $|s| \to 0$ and simultaneously $|z| \to +\infty$, in such a way that the product sz^2 remains equal to a complex constant $(\equiv \alpha^2/2)$, reduction to the corresponding *coherent* state is easily observed. In particular, equation (2) gives in this limiting case $a_n \to (\alpha^n / n!^{1/2}) \exp(-|\alpha|^2/2)$. Reduction to the *squeezed vacuum* state ($|z| \to 0$) is also easily observed.

Next, we turn our attention to the conditions which produce oscillations in the probability distribution

$$|a_n|^2=|N|^2\frac{|s|^n}{2^n n!}|H_n(z)|^2 \tag{9}$$

of the vibrational quanta of squeezed states. Firstly, we note that the oscillations (if any) are time independent. According to the semi-classical considerations presented in [8–10] the probability $|a_n|^2$ of finding n vibrational quanta in the squeezed state is governed by the overlap in the phase space between the elliptical band (with the area $2\pi\hbar$), representing the nth number state $u_n(x)$, and the squeezed-state uncertainty ellipse [13, 14]

$$\frac{|x-x_c(t)|^2}{(\delta x)^2}+\frac{|p-p_c(t)|^2}{(\delta p)^2}=1 \tag{10}$$

based on the joint x–p probability distribution (Wigner–Cohen function) $P_{sz}(x, p, t)=|\Psi_{sz}(x, t)|^2|\Phi_{sz}(p, t)|^2$ introduced in [15, 16]. Here, $\Phi_{sz}(p, t)$ denotes the corresponding momentum space wavefunction. The centre of the squeezed-state uncertainty ellipse is defined with

$$x_c(t)=\langle x\rangle=\frac{\delta\varepsilon+\delta^*\varepsilon^*}{\delta+\delta^*}=X\cos(\omega t-\varphi) \tag{11}$$

and

$$p_c(t)=\langle p\rangle=\frac{-2i\hbar|\delta|^2(\varepsilon-\varepsilon^*)}{\delta+\delta^*}=-m\omega X\sin(\omega t-\varphi). \tag{12}$$

The point $(x_c(t), p_c(t))$ follows classical motion of the harmonic oscillator with amplitude

$$X\equiv\frac{2^{3/2}\sigma|s|^{1/2}|z|}{1-|s|^2}(1+|s|^2-2|s|\cos 2\zeta)^{1/2} \tag{13}$$

and phaseshift

$$\tan\varphi\equiv\frac{\sin(\theta/2+\zeta)-|s|\sin(\theta/2-\zeta)}{\cos(\theta/2+\zeta)-|s|\cos(\theta/2-\zeta)}. \tag{14}$$

In (10), δx and δp denote the uncertainties

$$\delta x=\frac{1}{[2(\delta+\delta^*)]^{1/2}}=\sigma\left[\frac{1+|s|^2-2|s|\cos(\theta-2\omega t)}{1-|s|^2}\right]^{1/2} \tag{15}$$

$$\delta p=\frac{2^{1/2}\hbar|\delta|}{(\delta+\delta^*)^{1/2}}=\frac{\hbar}{2\sigma}\left[\frac{1+|s|^2+2|s|\cos(\theta-2\omega t)}{1-|s|^2}\right]^{1/2} \tag{16}$$

with

$$\sigma\equiv\left(\frac{\hbar}{2m\omega}\right)^{1/2} \tag{17}$$

denoting the halfwidth of the harmonic oscillator ground state $u_0(x)$. The phase space path, traced by the centre of the squeezed-state uncertainty ellipse, is therefore defined with

$$\frac{[x_c(t)]^2}{X^2}+\frac{[p_c(t)]^2}{(m\omega X)^2}=1. \tag{18}$$

Consider now, for example, the special case $\omega t-\varphi=2k\pi$, with k an integer. In order to get intersection (and, consequently, oscillations [8-10]) between the elliptical band, representing the nth number state $u_n(x)$, and the squeezed-state uncertainty ellipse, the conditions $X<2\sigma n^{1/2}$ and $P<(\delta p)_{\max}$ must be approximately satisfied. Here

$$P\equiv\frac{\hbar n^{1/2}}{\sigma}\left(1-\frac{X^2}{4\sigma^2 n}\right)^{1/2}. \tag{19}$$

Such considerations lead in this case (and in general as well, since the oscillations are time independent) to the condition $n_{\min}<n<n_{\max}$, with $n_{\min}\approx(X/2\sigma)^2$ and $n_{\max}\approx n_{\min}+(1+|s|)/(1-|s|)$ which must be fulfilled for oscillations in the probability distribution of excitation to appear. We found, however, that this is not sufficient; additionally, the phase ζ of the complex parameter z must be in the vicinity of $\zeta=0$ or $\zeta=\pi(|\mathrm{Im}\{z\}|\ll|z|)$. In such a case, $n_{\min}$ reduces to $n_{\min}\approx 2|s||z|^2/(1+|s|)^2$. Moreover, $n_{\min}\to|z|^2/2$ in the limit of large squeezing (when the squeeze parameter $|s|\to 1$). A generic example is depicted in figure 1. The observed dependence on ζ comes from the factor $|H_n(z)|^2$ in (9). Indeed, with the help of the explicit expression for $H_n(z)$ given in [17] we obtain, after some algebra,

$$|H_n(z)|^2=2(n!)^2\sum_{k=0}^{[n/2]}(-1)^k C_k^{(n)}\cos(2k\zeta) \tag{20}$$

with $[n/2]$ denoting the integer part of $n/2$ and with positive coefficients $C_k^{(n)}=C_k^{(n)}(|z|)$ defined via

$$C_k^{(n)}\equiv\frac{1}{1+\delta_{k0}}\sum_{l=0}^{[n/2]-k}\frac{(2|z|)^{2(n-k-2l)}}{l!(k+l)!(n-2l)!(n-2k-2l)!}. \tag{21}$$

In the case $\zeta=0$ or $\zeta=\pi$, the sum in (20) reduces to the alternating one $\Sigma_{k=0}^{[n/2]}(-1)^k C_k^{(n)}$. Because of delicate balancing of terms in this sum, an increase in the value of the vibrational quantum number n can (and usually does) result in a large change in its value leading to the observed oscillations. The factor $|N|^2\,(|s|^n/2^n n!)$ modulates and/or suppresses these oscillations (present, in principle, for any $|s|$). In the other extreme case, when $\zeta=\pi/2$ or $\zeta=3\pi/2$ the sum in (20) reduces simply to $\Sigma_{k=0}^{[n/2]}C_k^{(n)}$ leading to smooth and monotonic variation with n, and the oscillations are absent.

In conclusion, in this letter we have discussed the conditions for the existence of oscillations in the distribution of the vibrational quanta for the general case of the time-evolving squeezed state (which does not remain a minimum-uncertainty state) of the one-dimensional harmonic oscillator. In particular, we introduced a different parametrization of squeezed states and, with its help, found that in addition to the usual conditions which produce oscillations, the value of the phase ζ of the complex parameter z, which is the argument of the Hermite polynomial appearing in the expression for the expansion coefficient a_n of the squeezed state in the number state basis, is important. We found that, in order to obtain the oscillations, the phase must have a value in the vicinity of $\zeta=0$ or $\zeta=\pi$. We showed how this necessary condition results from the explicit expression for $|H_n(z)|^2$.

The oscillations discussed in this letter have been recognized as a striking feature of highly non-classical, squeezed states. To detect these states in the realm of quantum optics, using oscillatory counting distribution, it is important to know the conditions leading to the oscillations. From a wider perspective, it has become possible in recent years to almost perfectly isolate single quantum harmonic oscillators from their environment [18-19], thus enabling detailed investigations of the dynamics of this simplest

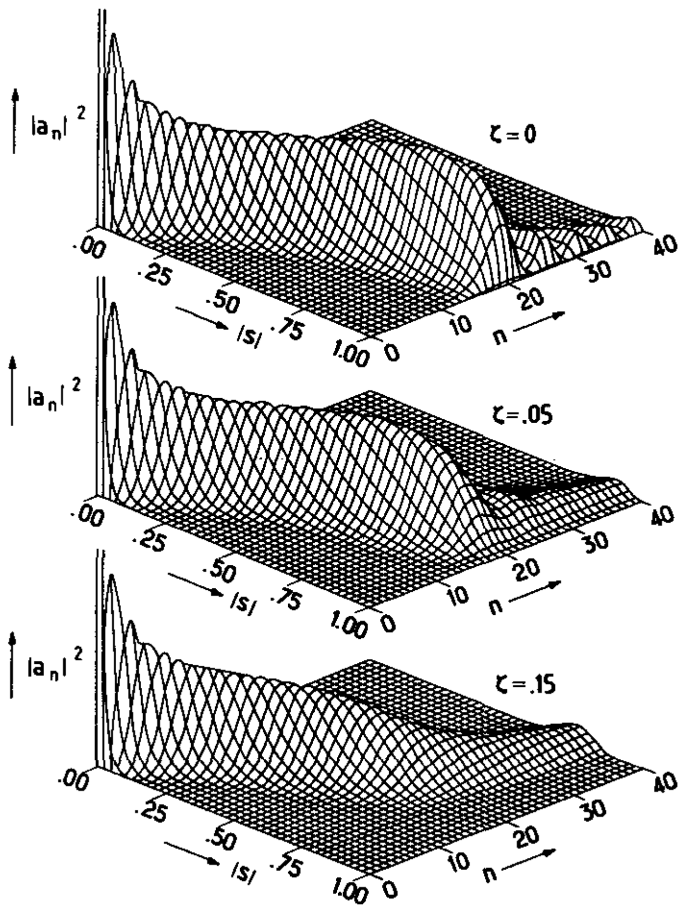

Figure 1. Probability $|a_n|^2$ of finding n vibrational quanta in a squeezed state for three different choices of the phase ζ indicated on the figure. All curves here are plotted, for definiteness, for the same value $|z|=6$. The probability is independent of time t and phase θ of the complex squeeze parameter s. As ζ increases from the zero value the oscillations rapidly vanish.

of all quantum systems. Such studies will, hopefully, yield a deeper understanding of quantum mechanics of single, isolated systems.

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